

# Vacuum static Brans-Dicke wormhole

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A simple Lorentzian vacuum wormhole solution of Brans-Dicke gravitation is presented and analysed. It is shown that such solution holds for both, the Brans-Dicke theory endowed with torsion (for a value of the coupling parameter  $\omega > 1/2$ ) and for the vacuum -no torsion- case (for  $\omega < -2$ ).  
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Since the renaissance, after the work of Morris and Thorne [1], the study of classical wormhole solutions have raised an enormous interest. Among the reasons that support this, one of them is the possibility of constructing time machines [2]. Another is related to the exoticity of the matter that threads this kind of geometry, which entails violations of the weak energy condition (WEC). It is curious that these two motivations also acts as the main theoretical problems that wormhole construction would have. A backward time travel would violate the chronology protection conjecture while exotic matter, with negative energy densities, have only appeared at quantum level, with no macroscopic analogy. In this scheme, studies of possible wormhole solutions in alternative gravitation was thought of as a way of understanding the role of WEC violation, together with the aim of finding phenomena for which different qualitative behaviors to those of General Relativity may arise.

In this report we are interested in Brans-Dicke gravitation [3]. This alternative theory, has proved to be a useful tool in the understanding of early universe models while providing with correct predictions of weak field tests and nucleosynthesis. The case for Lorentzian wormholes in Brans-Dicke theory has been analysed by Agnese and La Camera [4] and Nandi et. al. [5]. It was shown that three of the four Brans' classes of vacuum solutions admit a wormholelike spacetime for convenient choices of their parameters. A non-vacuum static solution was also presented where WEC was violated by the terms corresponding to the scalar, being matter non-exotic [6]. In addition, Euclidean wormholes have also been studied in this gravitational arena in [7].

In what follows, we shall present a static vacuum wormhole solution of the BD theory endowed with torsion [8]. This solution, previously discarded as a general static spherically symmetric one due to its inability in describing the whole range of the radial coordinate [9,10], is now reworked and analysed as a wormhole solution. Using afterwards the equivalence of the system of field equations between the torsion case and the vacuum one, we comment on the validity of this same solution for the case of a static vacuum BD wormhole, by suitably changing the BD coupling  $\omega$ . It is found that this solution is the most simple BD wormhole geometry found up to date. Its simplicity and well behaved asymptotic properties leads to the possibility of obtaining the dependence of the proper wormhole radial coordinate in a fully analytical fashion.

We begin introducing briefly the skeleton of the BD theory in a spacetime with torsion. In order to do so, let us first write down the action integral for the modified vacuum BD theory,

$$I = \int d^4x \sqrt{-g} \left( -\phi R + \omega \frac{\phi^\mu \phi_{;\mu}}{\phi} \right), \quad (1)$$

where  $\phi$  is the BD scalar and  $\omega$  is an undetermined constant. The scalar curvature is that of a Riemann-Cartan spacetime related to the tetrad field  $e_a^\mu$  and the spin connection  $\omega_{ab}^\mu$  by,

$$R = e_a^\mu e_b^\nu R_{\mu\nu}^{ab} = e_a^\mu e_b^\nu (\omega_{\mu,\nu}^{ab} - \omega_{\nu,\mu}^{ab} + \omega_{c\nu}^a \omega_{\mu}^{cb} - \omega_{c\mu}^a \omega_{\nu}^{cb}). \quad (2)$$

The torsion field is defined as

$$F^a_{\mu\nu} = e^a_{\mu,\nu} - e^a_{\nu,\mu} + \omega_{c\nu}^a e_{\mu}^c - \omega_{c\mu}^a e_{\nu}^c. \quad (3)$$

Henceforth we shall use the conventions of [9] unless otherwise specified. The field equations that follow from such an action are given as,

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$$R_{\mu\nu} = \omega \phi_{,\mu} \phi_{,\nu} \phi^{-2}, \quad (4a)$$

$$F^\mu{}_{\alpha\beta} = \frac{1}{2} \left( \phi_{,\alpha} \delta^\mu_\beta - \phi_{,\beta} \delta^\mu_\alpha \right) \phi^{-1}, \quad (4b)$$

$$\Delta\phi = F^\mu{}_{\lambda\mu} \phi^{,\lambda}. \quad (4c)$$

At this stage, and in the sake of completeness, it is important to review the basic properties a spacetime needs to obey so as to display wormholelike features. We begin by introducing the static spherically symmetric line element,

$$ds^2 = -e^{2\Phi(r)} dt^2 + e^{2\Lambda(r)} dr^2 + r^2 d\vartheta^2 + r^2 \sin^2 \vartheta d\varphi^2 \quad (5)$$

where the redshift function  $\Phi$  and the shapelike function  $e^{2\Lambda}$  characterize the wormhole topology and must satisfy,  
*i)*  $e^{2\Lambda} \geq 0$  throughout the spacetime. This is required to ensure the finiteness of the proper radial distance defined by  $dl = \pm e^\Lambda dr$ . The  $\pm$  signs refer to the two asymptotically flat regions which are connected by the wormhole throat.

*ii)* The precise definition of the wormhole's throat (minimum radius) entails a vertical slope of the embedding surface,

$$\lim_{r \rightarrow r_{\text{th}}^+} \frac{dz}{dr} = \lim_{r \rightarrow r_{\text{th}}^+} \pm \sqrt{e^{2\Lambda} - 1} = \infty. \quad (6)$$

*iii)* As  $l \rightarrow \pm\infty$  (or equivalently,  $r \rightarrow \infty$ ),  $e^{2\Lambda} \rightarrow 1$  and  $e^{2\Phi} \rightarrow 1$ . This is the asymptotic flatness condition on the wormhole spacetime.

*iv)*  $\Phi(r)$  needs to be finite throughout the spacetime to ensure the absence of event horizons and singularities.

*v)* Finally, the *flaring out* condition, that asserts that the inverse of the embedding function  $r(z)$ , must satisfy  $d^2r/dz^2 > 0$  at or near the throat. Stated mathematically,

$$-\frac{\Lambda' e^{-2\Lambda}}{(1 - e^{-2\Lambda})^2} > 0. \quad (7)$$

We turn back our attention to the previously posed problem. Replacing the line element in the field equations, the non-trivial set is

$$\left( \Phi' + \frac{\phi'}{2\phi} \right)' + \left( \Phi' + \frac{\phi'}{2\phi} \right) \left( \Phi' - \Lambda' + \frac{\phi'}{\phi} + \frac{2}{r} \right) = 0 \quad (8)$$

$$(1 + \Phi' - \Lambda') \left( \Phi' + \frac{\phi'}{2\phi} \right) + (1 - \Lambda') \left( \frac{2}{r} + \frac{\phi'}{\phi} \right) - \omega \left( \frac{\phi'}{\phi} \right)^2 = 0 \quad (9)$$

$$\left( \frac{1}{r} + \frac{\phi'}{2\phi} \right) \left[ \Phi' - \Lambda' + \frac{1}{r} \left( \frac{1}{r} + \frac{\phi'}{\phi} \right) \right] + \left( \frac{1}{r} + \frac{\phi'}{2\phi} \right)' - \frac{e^{2\Lambda}}{r} = 0, \quad (10)$$

$$\frac{\phi''}{\phi'} = (\Lambda' - \Phi') - \frac{2}{r}. \quad (11)$$

As was shown in [9], combining the field equations and with a suitable choice of the integration constants, the redshift and the shapelike functions could be linked to the BD scalar as follows,

$$\Phi' = -\frac{\phi'}{\phi} \quad e^{2\Lambda} = \alpha \left( \frac{\phi'}{\phi} \right)^2 r^4 \quad (12)$$

with  $\phi'/\phi = \pm \sqrt{\alpha r^2 - \varrho}$ , being  $\alpha$  and  $\varrho$  positive constants<sup>1</sup>. A trivial integration yields

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<sup>1</sup>Note that  $\varrho = (1 - 2\omega)/4$  could be related to the parameter  $\beta^2$  of [9].

$$\phi = \phi_0 \exp \left[ \frac{1}{\sqrt{8}} \arcsen \left( -\sqrt{\frac{\varrho}{r^2 \alpha}} \right) \right] \quad (13)$$

$$e^{2\Lambda} = \frac{\alpha r^2}{\alpha r^2 - \varrho} \quad (14)$$

$$e^{2\Phi} = \exp \left\{ -2 \left[ \frac{1}{\sqrt{\varrho}} \arcsen \left( -\sqrt{\frac{\varrho}{r^2 \alpha}} \right) + \mathcal{K} \right] \right\}. \quad (15)$$

We shall put into evidence that this solution, valid whenever  $r \geq \sqrt{\varrho/\alpha}$ , and  $\omega > 1/2$ , satisfies the desired properties of traversable wormholes. It is easily seen that in the asymptotic limit, the redshift and shapelike functions tends to unity, being the metric tensor that of a flat Minkowskian spacetime. Besides, the BD scalar also tends to unity if  $\mathcal{K} = -\pi/\sqrt{\varrho}$ . Finally, at the throat, when  $r \rightarrow \sqrt{\varrho/\alpha}$ ,  $dz/dr \rightarrow \infty$ .

As the  $r$  coordinate system has a singularity at the throat, where the metric coefficient  $g_{rr}$  becomes divergent, the spatial geometry is better studied in terms of the aforementioned proper radial coordinate  $l$ , which can be computed as stated in property *i*). This yields

$$l = \sqrt{r^2 - \frac{\varrho}{\alpha}}. \quad (16)$$

Due to the simple expression for  $l(r)$  it is easy to rewrite the metric tensor in terms of this proper radial distance,

$$ds^2 = -h(l) dt^2 + dl^2 + r^2(l) d\Omega^2, \quad (17)$$

where,

$$h(l) = \exp \left\{ -2 \left[ \frac{1}{\sqrt{\varrho}} \arcsen \left( \sqrt{\frac{\varrho}{\alpha l^2 + \varrho}} \right) + \mathcal{K} \right] \right\}. \quad (18)$$

and

$$r^2(l) = l^2 + \frac{\varrho}{\alpha}. \quad (19)$$

Thus, in this well behaved coordinate system, as  $l$  increases from  $-\infty$  to 0,  $r$  decreases monotonically to a minimum value at the throat; and as  $l$  increases onwards to  $+\infty$ ,  $r$  increases monotonically.

As a final remark, we would wish to note that the field equations of the torsion endowed BD case are equivalent to those of vacuum [9,10] by making the substitution  $\omega \rightarrow -\omega - 3/2$ . This immediately implies that the solution presented is also a vacuum spherical symmetric solution of Brans-Dicke theory without torsion with  $\omega < -2$ . In this case it was previously noted that the stress-energy tensor of the BD field is WEC violating, being the field itself the carrier of exoticity. That was the case found in the works of Agnese and La Camera [4] and Nandi et. al. [5] for others vacuum solutions and also in [6] for a non-vacuum case. Thus, wormhole taxonomy is now enriched with a solution that, being discarded in the past as bad behaved, is now recasted as the most simple example of wormhole geometry in alternative gravity.

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